GENERALIZED (σ,τ)-DERIVATIONS IN PRIME RINGS

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ABSTRACT: Let R be a prime ring, I be a non-zero ideal of R. Suppose that $F: R \to R$ be a generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation $g: R \to R$ respectively and $\tau(I) \neq 0$.In this paper, we studied the following identities in prime rings: (i) $F([x,y]) = \pm\sigma(xy \pm yx)$; (ii) $F((xoy)) = \pm\sigma(xy \pm yx)$; (iii) $[F(x),y]_{\sigma,\tau} \pm [G(y),x]_{\sigma,\tau} = 0$for all x,y in some suitable sub sets of R.

Keywords: Prime ring, Derivation, Generalized derivation, (σ,τ)-derivation, Generalized (σ,τ)-derivation

Introduction: Bresar in [2], first time introduced the notion of generalized derivation. In 1992, Daif et al. in [4], proved a result which is given as let R be a semiprime ring, I be a non zero ideal of R and d be a derivation on R such that $d([x,y]) = [x,y]$, for all $x,y \in I$, then $I \subseteq Z(R)$. In 2002, Ashraf and Rehman [1] extended the result of Daif et al. [4] by replacing ideal to lie ideal. In 2003, Quadri et al. in [7] extended the result of Ashraf et al. [1] on generalized derivation given as let R be a prime ring with characteristic different from two, I be a nonzero ideal of R and F be a generalized derivation on R associated with a derivation d on R such that $F([x,y]) = [x,y]$, for all $x,y \in I$, then R is commutative. Golbasi et al. in [5] extended the result of Quadri et al. [7] by replacing ideal to lie ideal. Recently, S.K. Tiwari et al. in [8] studied Multiplicative (generalized)-derivation in semiprime rings. Further Chirag Garg et al. in [3] studied on generalized (α,β)-derivations in prime rings. In this paper we extended of S.K. Tiwari et al. in [8], we have proved some results on generalized (σ,τ)-derivations in prime rings.

Preliminaries: Throughout this paper R denote an associative ring with center Z. Recall that a ring R is prime if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. For any $x,y \in R$, the symbol $[x,y]$ stands for the commutator $xy - yx$ and the symbol (xoy) denotes the anticommutator $xy + yx$. Let σ,τ be any two automorphisms of R. For any $x,y \in R$, we set $[x,y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$ and $(xoy)_{\sigma,\tau} = x\sigma(y) + \tau(y)x$. An additive mapping $d: R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x,y \in R$. An additive mapping $d: R \to R$ is called a (σ,τ)-derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ holds for all $x,y \in R$. An additive mapping $F: R \to R$ is called a generalized derivation, if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x,y \in R$. An additive mapping $F: R \to R$ is said to be a generalized (σ,τ)-derivation of R, if there exists a (σ,τ)-derivation $d: R \to R$ such that $F(xy) = F(x)\sigma(y) + \tau(x)d(y)$ holds for all $x,y \in R$.

Throughout this paper, we shall make use of the basic commutator identities:

$[x, yz] = y[x, z] + [x, y]z,$

$[xy,z] = [x,z]y + x[y, z],$

$(xo(yz)) = (xoy)z - y[x,z] = y(xoz) + [x, y]z,$

$[xy,z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x,\tau(z)]y = x[y,\sigma(z)] + [x,z]_{\sigma,\tau}y,$

$[x, yz]_{\sigma,\tau} = \tau(y)[x,z]_{\sigma,\tau} + [x,y]_{\sigma,\tau}\sigma(z),$

$(xo(yz))_{\sigma,\tau} = (xoy)_{\sigma,\tau}\sigma(z) - \tau(y)[x,z]_{\sigma,\tau} = \tau(y)(xoz)_{\sigma,\tau} + [x,y]_{\sigma,\tau}\sigma(z),$

$((xy)oz)_{\sigma,\tau} = x(yoz)_{\sigma,\tau} - [x,\tau(z)]y = (xoz)_{\sigma,\tau}y + x[y,\sigma(z)].$

Lemma 1[6, Lemma 1.1]: Let R be a prime ring with characteristics not two and U a non zero lie ideal of R. If d is a non zero (σ,τ)- derivation of R such that $d(U) = O$, then $U \subseteq Z$.

Theorem 1: Let R be a prime ring, I be a non-zero ideal of R. Suppose that $F{:}\,R \to R$ be a generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation $g{:}\,R \to R$ respectively and $\tau(I) \neq 0$. If $F([x,y]) = \pm\sigma(xy \pm yx)$, for all $x, y \in I$, then either, $[g(x),x] = 0$, for all $x \in I$ or R is commutative.

Proof: By the assumption, we have $F([x,y]) = \pm\sigma(xy \pm yx)$, for all $x, y \in I$. (1)

We replacing y by yx in equation (1), we obtain

$F([x,yx]) = \pm\sigma(xyx \pm yxx)$, for all $x, y \in I$

$F([x,y]x + y[x,x]) = \pm\sigma(xy \pm yx)\sigma(x)$

$F([x,y]x) = \pm\sigma(xy \pm yx)\sigma(x)$

$F([x,y])\sigma(x) + \tau[x,y]g(x) = \pm\sigma(xy \pm yx)\sigma(x)$

Using equation (1), it reduces to

$F([x,y])\sigma(x) + \tau([x,y])g(x) = F([x,y])\sigma(x)$, for all $x, y \in I$.

$\tau([x,y])g(x) = 0$, for all $x, y \in I$. (2)

We replacing y by $g(x)y$ in equation (2), we get

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$\tau([x, g(x)y])g(x) = 0$

$\tau([x, g(x)])\tau(y)g(x) + \tau(g(x))\tau([x,y])g(x) = 0$, for all $x, y \in I$. (3)

Left multiplying equation (2) by $\tau(g(x))$, we get

$\tau(g(x))\tau[x, y]g(x) = 0$, for all$x, y \in I$. (4)

We subtracting equation (3) from equation (4), we get

$\tau([g(x),x])\tau(y)g(x) = 0$, for all $x, y \in I$. (5)

We replacing y by ry, $r \in R$ in the equation (5), we get

$\tau([g(x),x])\tau(ry)g(x) = 0$

$\tau([g(x),x])\tau(r)\tau(y)g(x) = 0$

$\tau([g(x),x])R\tau(y)g(x) = 0$, for all $x, y \in I$.

Since R is prime, we get either $\tau([g(x),x]) = 0$, for all $x \in I$ or $\tau(y)g(x) = 0$, for all $x, y \in I$.

Since τ is an automorphism of R and $\tau(I) \neq 0$, we have either $[g(x),x] = 0$, for all $x \in I$ or $g(x) = 0$, for all $x \in I$.

Now let $A = \{x \in I/[g(x),x] = 0\}$ and $B = \{x \in I/g(x) = 0\}$.

Clearly, A and B are additive proper subgroups of I whose union is I.

Since a group cannot be the set theoretic union of two proper subgroups.

Hence either $A = I$ or $B = I$.

If $A = I$, then $[g(x),x] = 0$, for all $x \in I$.

On the other hand, if $B = I$, then $g(x) = 0$, for all $x \in I$, by lemma 1 implies that R is commutative.

Theorem 2: Let R be a prime ring, I be a non-zero ideal of R. Suppose that $F: R \rightarrow R$ be a generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation $g: R \rightarrow R$ respectively and $\tau(I) \neq 0$. If $F((xoy)) = \pm\sigma(xy \pm yx)$, for all $x, y \in I$, then either, $[g(x),x] = 0$, for all $x \in I$ or R is commutative.

Proof: By the assumption, we have $F((xoy)) = \pm\sigma(xy \pm yx)$, for all $x, y \in I$. (6)

We replacing y by yx in equation (6), we obtain

$F((xoyx)) = \pm\sigma(xyx \pm yxx)$, for all $x, y \in I$

$F\big((xoy)x - y[x,x]\big) = \pm\sigma(xy \pm yx)\sigma(x)$

$F((xoy)x) = \pm\sigma(xy \pm yx)\sigma(x)$

$F((xoy))\sigma(x) + \tau((xoy))g(x) = \pm\sigma(xy \pm yx)\sigma(x)$

Using equation (6), it reduces to

$F((xoy))\sigma(x) + \tau((xoy))g(x) = F((xoy))\sigma(x)$, for all $x, y \in I$.

$\tau((xoy))g(x) = 0$, for all $x, y \in I$. (7)

We replacing y by $g(x)y$ in equation (7), we get

$\tau((xog(x)y))g(x) = 0$

$\tau(g(x)(xoy) + [x, g(x)]y)g(x) = 0$,

$\tau(g(x))\tau\big((xoy)\big)g(x) + \tau([x,g(x)])\tau(y)g(x) = 0$,for all $x, y \in I$. (8)

Left multiplying equation (7) by $\tau\big(g(x)\big)$, we get

$\tau\big(g(x)\big)\tau((xoy))g(x) = 0$, for all $x, y \in I$. (9)

We subtracting equation (8) from equation (9), we get

$\tau([g(x),x])\tau(y)g(x) = 0$, for all $x, y \in I$. (10)

The equation (10) is same as equation (5) in theorem 1. Thus, by same argument of theorem 1, we can conclude the result are $[g(x), x] = 0$, for all $x \in I$ or R is commutative.

Theorem 3: Let R be a prime ring and I be a non-zero ideal on R. Suppose that G and F are two generalized (σ,τ)-derivation on R associated with (σ,τ)-derivation g and d on R respectively and $\tau(I) \neq 0$. If $[F(x), y]_{\sigma,\tau} \pm [G(y), x]_{\sigma,\tau} = 0$, for all $x, y \in I$, then either $[d(x), x] = 0$, $[g(x), x] = 0$, for all $x \in I$ or R is commutative.

Proof: First we consider the case $[F(x), y]_{\sigma,\tau} + [G(y), x]_{\sigma,\tau} = 0$, for all $x, y \in I$. (11)
We replacing y by yx in equation (11), we obtain

$[F(x), yx]_{\sigma,\tau} + [G(yx), x]_{\sigma,\tau} = 0$, for all $x, y, z \in I$

$$[F(x),y]_{\sigma,\tau}\sigma(x)+\tau(y)[F(x),x]_{\sigma,\tau}+[G(y)\sigma(x)+\tau(y)g(x),x]_{\sigma,\tau}=0$$

$$[F(x),y]_{\sigma,\tau}\sigma(x)+\tau(y)[F(x),x]_{\sigma,\tau}+[G(y)\sigma(x),x]_{\sigma,\tau}+[\tau(y)g(x),x]_{\sigma,\tau}=0$$

$$[F(x),y]_{\sigma,\tau}\sigma(x)+\tau(y)[F(x),x]_{\sigma,\tau}+[G(y),x]_{\sigma,\tau}\sigma(x)+G(y)[\sigma(x),\sigma(x)]+\tau(y)[g(x),x]_{\sigma,\tau}+ [\tau(y),\tau(x)]g(x)=0$$

$$([F(x),y]_{\sigma,\tau}+[G(y),x]_{\sigma,\tau})\sigma(x)+\tau(y)[F(x),x]_{\sigma,\tau}+\tau(y)[g(x),x]_{\sigma,\tau}+[\tau(y),\tau(x)]g(x)=0$$

Using equation (11), it reduces to

$\tau(y)[F(x),x]_{\sigma,\tau}+\tau(y)[g(x),x]_{\sigma,\tau}+[\tau(y),\tau(x)]g(x)=0$, for all $x,y\in I$. (12)

We replacing y by $g(x)y$ in equation (12), we get

$\tau(g(x)y)[F(x),x]_{\sigma,\tau}+\tau(g(x)y)[g(x),x]_{\sigma,\tau}+[\tau(g(x)y),\tau(x)]g(x)=0$, for all $x,y\in I$.

$$\tau(g(x))\tau(y)[F(x),x]_{\sigma,\tau}+\tau(g(x))\tau(y)[g(x),x]_{\sigma,\tau}+[\tau(g(x))\tau(y),\tau(x)]g(x)=0$$

$$\tau(g(x))\tau(y)[F(x),x]_{\sigma,\tau}+\tau(g(x))\tau(y)[g(x),x]_{\sigma,\tau}+\tau(g(x))[\tau(y),\tau(x)]g(x)+ [\tau(g(x)),\tau(x)]\tau(y)g(x)=0$$

, for all $x,y\in I$. (13)

Left multiplying equation (12) by $\tau(g(x))$, we get

$\tau(g(x))\tau(y)[F(x),x]_{\sigma,\tau}+\tau(g(x))\tau(y)[g(x),x]_{\sigma,\tau}+\tau(g(x))[\tau(y),\tau(x)]g(x)=0$, for all $x,y\in I$. (14)

We subtracting equation (14) from equation (13), we get

$$[\tau(g(x)),\tau(x)]\tau(y)g(x)=0,$$

$\tau([g(x),x])\tau(y)g(x)=0$, for all $x,y\in I$. (15)

The equation (15) is same as equation (5) in theorem 1. Thus, by same argument of theorem 1, we can conclude the result are $[g(x),x]=0$, for all $x\in I$ or R is commutative.

Again we replacing x by xy in equation (11), we get

$[F(xy),y]_{\sigma,\tau}+[G(y),xy]_{\sigma,\tau}=0$, for all $x,y,z\in I$

$$[F(x)\sigma(y)+\tau(x)d(y),y]_{\sigma,\tau}+[G(y),x]_{\sigma,\tau}\sigma(y)+\tau(x)[G(y),y]_{\sigma,\tau}=0$$

$$[F(x)\sigma(y),y]_{\sigma,\tau}+[\tau(x)d(y),y]_{\sigma,\tau}+[G(y),x]_{\sigma,\tau}\sigma(y)+\tau(x)[G(y),y]_{\sigma,\tau}=0$$

$[F(x),y]_{\sigma,\tau}\sigma(y)+F(x)[\sigma(y),\sigma(y)]+\tau(x)[d(y),y]_{\sigma,\tau}+[\tau(x),\tau(y)]d(y)+[G(y),x]_{\sigma,\tau}\sigma(y)+\tau(x)[G(y),y]_{\sigma,\tau}=0$

$([F(x),y]_{\sigma,\tau}+[G(y),x]_{\sigma,\tau})\sigma(y)+\tau(x)[d(y),y]_{\sigma,\tau}+[\tau(x),\tau(y)]d(y)+\tau(x)[G(y),y]_{\sigma,\tau}=0$

Using equation (11), it reduces to

$\tau(x)[d(y),y]_{\sigma,\tau}+[\tau(x),\tau(y)]d(y)+\tau(x)[G(y),y]_{\sigma,\tau}=0$, for all $x,y\in I$. (16)

We replacing x by $d(y)x$ in equation (16), we get

$\tau(d(y)x)[d(y),y]_{\sigma,\tau}+[\tau(d(y)x),\tau(y)]d(y)+\tau(d(y)x)[G(y),y]_{\sigma,\tau}=0$, for all $x,y\in I$.

$\tau(d(y))\tau(x)[d(y),y]_{\sigma,\tau}+[\tau(d(y))\tau(x),\tau(y)]d(y)+\tau(d(y))\tau(x)[G(y),y]_{\sigma,\tau}=0$
$\tau\big(d(y)\big)\tau(x)[d(y),y]_{\sigma,\tau}+\tau\big(d(y)\big)[\tau(x),\tau(y)]d(y)+[\tau(d(y)),\tau(y)]\tau(x)d(y)+\tau(d(y))\tau(x)[G(y),y]_{\sigma,\tau}=0$
, for all $x,y\in I$. (17)

Left multiplying equation (16) by $\tau(d(y))$, we get

$\tau\big(d(y)\big)\tau(x)[d(y),y]_{\sigma,\tau}+\tau\big(d(y)\big)[\tau(x),\tau(y)]d(y)+\tau(d(y))\tau(x)[G(y),y]_{\sigma,\tau}=0$, for all $x,y\in I$.
(18) We subtracting equation (18) from equation (17), we get

$\big[\tau\big(d(y)\big),\tau(y)\big]\tau(x)d(y)=0$, for all $x,y\in I$.

We replacing x by y and y by x in the above equation, we get

$\tau([d(x),x])\tau(y)d(x)=0$, for all $x,y\in I$. (19)

The equation (19) is same as similar equal equation (5) in theorem 2. Thus, by same argument of theorem 2, we can conclude the result are $[d(x),x]=0$, for all $x\in I$ or R is commutative. Using similar approach we conclude that the same results holds for $[F(x),y]_{\sigma,\tau}-[G(y),x]_{\sigma,\tau}=0$, for all $x,y\in I$.

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